Noise Induced Phase Synchronization of a General Class of Limit Cycle Oscillators

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We consider the effect of weak additive noise to a general class of limit cycle oscillators. Employing phase reduction method, the largest Lyapunov exponent of the noise driven oscillators are shown to be negative for a broad class of oscillators, which implies that identical limit cycle oscillators driven by a common additive noise can achieve phase synchronization without direct mutual interactions. Generalization of the results including the effect of colored noise and the effect of noise to spatially extended systems are also discussed.

§1. Introduction

Phase synchronization is one of the most interesting phenomena of populations of nonlinear self-sustained oscillators.[1)] Phases of the oscillators can precisely coincide with each other owing to mutual interactions or external periodic forcing. Fluctuating inputs, whereas they disturb periodic oscillations, can also lead phase synchrony to self-sustained oscillators.[2)–5)] When two or more identical oscillators are driven by a common fluctuating input, their phases can synchronize. Evidence of the noise induced phase synchronization is accumulating in a variety of fields.[6),7)] The synchronization is characterized by the negative largest Lyapunov exponent of the driven systems. In this study, using phase reduction method[1)] which is applicable to an arbitrary limit cycle oscillators, we prove that the largest Lyapunov exponent of oscillators driven by weak noise are always negative for a broad class of oscillators.[3)] We also apply the universal result to spatially extended dynamical systems exhibiting spatio-temporal instabilities.

§2. Noise induced phase synchronization

Population of N identical nonlinear oscillators driven by common additive noise are described as

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + \boldsymbol{\xi}(t) \ , \tag{2·1}$$

where $i = 1, \cdots, N$ and $\boldsymbol{\xi}(t)$ is a vector of Gaussian white noise. The elements of the vector are normalized as $\langle \xi_l(t) \rangle = 0$ and $\langle \xi_l(t)\xi_m(s) \rangle = 2D_{lm}\delta(t-s)$, where $\boldsymbol{D} = (D_{lm})$ is a covariance matrix of the noise components. Regarding the common noise as a weak perturbation to the deterministic oscillators, the phase reduction method gives the following Stratonovich stochastic differential equation of the phases:

$$\dot{\phi}_i = \omega + \boldsymbol{Z}(\phi_i) \cdot \boldsymbol{\xi} \ , \tag{2·2}$$

where ω is an intrinsic frequency of the unperturbed oscillators. $\boldsymbol{Z}$ is the phase-dependent sensitivity defined as $\boldsymbol{Z}(\phi) = \mathrm{grad}_{\boldsymbol{X}}\phi|_{\boldsymbol{X}=\boldsymbol{X}_0(\phi)}$, where $\boldsymbol{X}_0(\phi)$ is the unperturbed limit cycle solution. Because all the oscillators are identical and do not interact with one another, we can study the phase synchronization of the entire population by analyzing linear stability of single oscillator of them. Linearization of the oscillator for a small deviation ψ gives

$$\dot{\psi} = (\boldsymbol{Z}'(\phi) \cdot \boldsymbol{\xi})\psi \ . \tag{2·3}$$

Since the largest Lyapunov exponent λ is defined as the long time average of the exponential growth rate of ψ, we can represent λ as

$$\begin{aligned} \lambda &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \boldsymbol{Z}'(\phi(t)) \cdot \boldsymbol{\xi}(t) dt \\ &= \langle \boldsymbol{Z}'(\phi(t)) \cdot \boldsymbol{\xi}(t) \rangle_\xi \\ &= \int_0^{2\pi} P_{st}(\phi)[(\boldsymbol{Z}'^T \boldsymbol{D} \boldsymbol{Z})' - (\boldsymbol{Z}'^T \boldsymbol{D} \boldsymbol{Z}')] d\phi \ . \end{aligned} \tag{2·4}$$

Here we replaced the long time average with the ensemble average with respect to ξ. P_{st} is a steady distribution function of ϕ. Under assumption of weak noise, P_{st} is reduced to a constant function, and we finally obtain the following formula:

$$\lambda = -\frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{Z}'^T \boldsymbol{D} \boldsymbol{Z}' d\phi < 0 \ . \tag{2·5}$$

This implies that weak noise always induce synchronization to limit cycle oscillators.

§3. Colored Gaussian noise

We have assumed in the above analysis that the driving noise $\boldsymbol{\xi}$ is a white Gaussian noise. Here we treat the case where weak driving noise has finite temporal correlation. To simplify, we assume there is only single noise source, it is straightforward to generalize the result to multiple noise sources. Phase reduction method is also valid even if ξ has temporal correlation. Since noise strength is small, we can approximate distributions of the phase as $P(\phi) \cong 1/(2\pi)$ and $P(\phi(t')|\phi(t)) \cong \delta(\phi(t') - \phi(t) + \omega(t - t'))$. Employing these approximations, we obtain the expression of the largest Lyapunov exponent as

$$\lambda = \frac{1}{2\pi} \int_0^\infty ds C(s) \int_0^{2\pi} d\phi Z''(\phi) Z(\phi - \omega s) \ . \tag{3·1}$$

We can prove the exponent is always negative by rewritten Eq. (3·1) using Fourier mode of Z and ξ:

$$\lambda = -\sum_{n=-\infty}^{\infty} n^2 |Z_n|^2 \frac{\langle |\xi_{n\omega}|^2 \rangle}{2} < 0 \ , \tag{3·2}$$

where $Z(\phi) = \sum_{n=-\infty}^{\infty} Z_n e^{in\phi}$ and $\xi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \xi_k e^{ikt} dk$. To confirm the result, we measured the Lyapunov exponent of a specific case by numerical simulation [Fig. 2]. Measured exponents well agree with the analytical result.

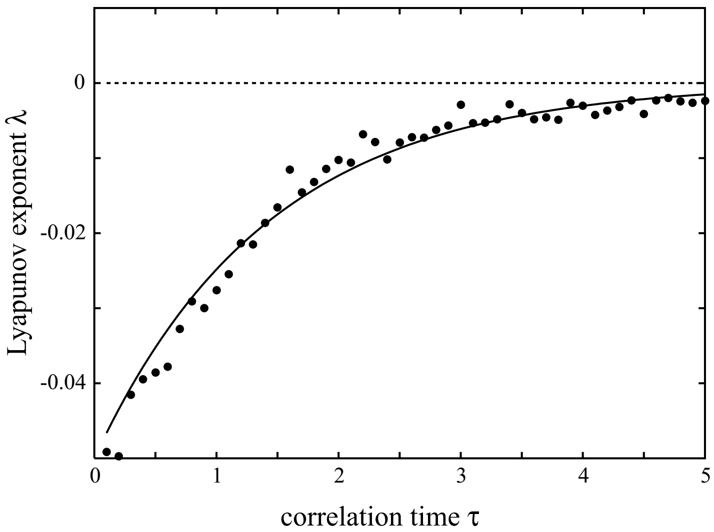

Fig. 1. Lyapunov exponents as a function of correlation time. Here $\boldsymbol{Z} \cdot \boldsymbol{\xi} = sin(\phi)\xi$ and $\langle \xi(t+s)\xi(t) \rangle = \frac{D}{\tau} e^{-\frac{s}{\tau}}$. The solid line shows an analytical result.

§4. Noise induced suppression of phase turbulence

Our analytical approach based on phase reduction method is also applicable to reaction-diffusion systems which may show spatio-temporal chaos. Here, we show that spatially uniform noise applying to reaction-diffusion systems generally can suppress phase turbulence.

Reaction-diffusion system with common additive noise are described as

$$\frac{\partial \boldsymbol{X}}{\partial t}(x,t) = \boldsymbol{F}(\boldsymbol{X}(x,t)) + \nabla^2 \boldsymbol{X} + \boldsymbol{\xi}(t) \ . \tag{4·1}$$

When $\boldsymbol{F}(\boldsymbol{X})$ has a limit cycle solution, uniform noisy oscillating state $\frac{\partial \boldsymbol{X}_0}{\partial t} = \boldsymbol{F}(\boldsymbol{X}_0) + \boldsymbol{\xi}(t)$ is a solution of Eq. (4·1). Using phase reduction, linear stability eigenvalues of the uniform state are obtained as a function of perturbation wave number k:

$$\lambda(k) = -\alpha k^2 - \gamma k^4 - \frac{1}{2\pi}\int_0^{2\pi} D Z'^2 d\phi \ . \tag{4·2}$$

In absent of additive noise, uniform oscillation is unstable and the system exhibits phase turbulence, if $\alpha < 0$. However, since last term of the right hand side is negative, there exists a critical strength of noise

$$D_c = \frac{\alpha^2}{4\gamma \bar{Z'^2}} \ , \tag{4·3}$$

above which $\lambda(k) < 0$ for any k and the uniform state recover linear stability. To demonstrate the result, we calculated Ginzburg-Landau equation driven by common additive noise numerically and found that spatio-temporal turbulence is actually suppressed by additive noise when the noise strength is sufficiently large.

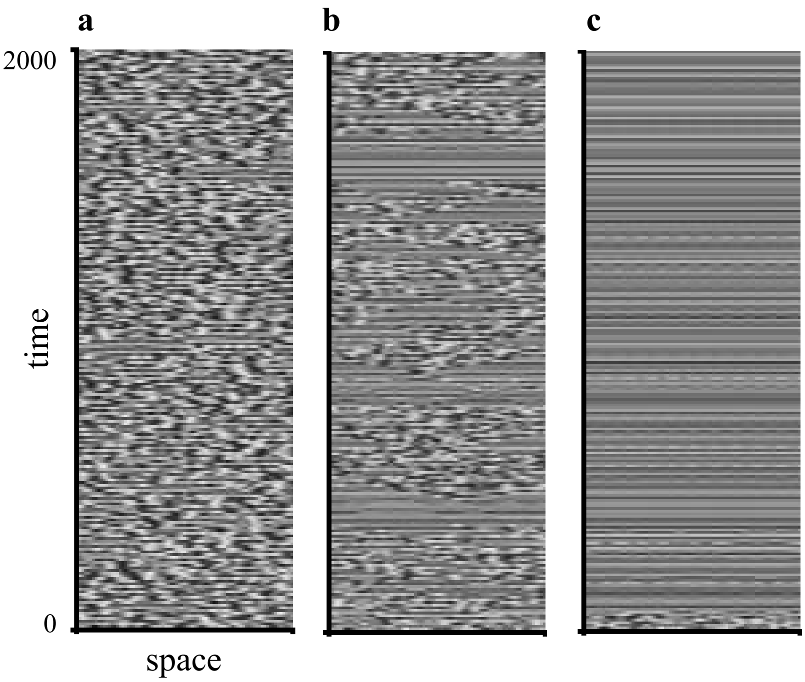

Fig. 2. Spatio-temporal profiles of complex Ginzburg-Landau equation with common additive noise, $\dot{A} = (1 + ic_0)A + (1 + ic_1)\nabla^2 A - (a + ic_2)|A|^2 A + \xi(t)$, at $c_0 = -1$, $c_1 = 1$ and $c_2 = -1.5$. (a) $D = 0.0$, (b) $D = 0.008$ and (c) $D = 0.013$. Critical noise strength evaluated from Eq. (4·3) is $D_c = 0.0118$.

In conclusion, we have found that limit cycle oscillators always synchronize in phase when they are driven by common weak additive noise. Analytical expression of the largest Lyapunov exponent of the driven oscillator has been derived, which has shown to be negative regardless of the detailed oscillatory dynamics and temporal correlation of the noise. We also found weak noise applying to spatially extended systems can suppress phase turbulence generally when the noise strength is above a critical strength.

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References

1) Y. Kuramoto, *Chemical Oscillation, Waves, and Turbulence* (Springer-Verlag, Tokyo, 1984); (Dover Edition, 2003).
2) A. S. Pikovsky, in *Nonlinear and Turbulent Processes in Physics*, ed. R. Z. Sagdeev (Harwood, Singapore, 1984), p. 1601.
3) J. Teramae and D. Tanaka, Phys. Rev. Lett. **93** (2004), 204103.
4) K. Nagai, H. Nakao and Y. Tsubo, Phys. Rev. E **71** (2005), 036217.
5) D. S. Goldobin and A. Pikovsky, Phys. Rev. E **71** (2005), 045201; Physica A **351** (2005), 126.
6) Z. F. Mainen and T. J. Sejnowski, Science **268** (1995), 1503.
7) L. Yu, E. Ott and Q. Chen, Phys. Rev. Lett. **65** (1990), 2935.