Dynamics of Limit-Cycle Oscillators Subject to General Noise

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The phase description is a powerful tool for analyzing noisy limit-cycle oscillators. The method, however, has found only limited applications so far, because the present theory is applicable only to Gaussian noise while noise in the real world often has non-Gaussian statistics. Here, we provide the phase reduction method for limit-cycle oscillators subject to general, colored and non-Gaussian, noise including a heavy-tailed one. We derive quantifiers like mean frequency, diffusion constant, and the Lyapunov exponent to confirm consistency of the results. Applying our results, we additionally study a resonance between the phase and noise.

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Limit-cycle oscillators effectively model various sustained oscillations in many fields of science and technology including chemical reactions, biology, electric circuits, and lasers [1–4]. The phase reduction method is a powerful analytical tool which approximates high-dimensional dynamics of limit-cycle oscillators with a single phase variable that characterizes timing of oscillation [1,5]. Since the phase is neutrally stable, phase perturbations persist in time and result in various remarkable phenomena where weak action leads to significant effects, such as those addressed in the theory of synchronization [6,7]. While the theory of the phase reduction had been developed for deterministic oscillators, recent studies successfully extended the theory to limit-cycle oscillators subject to noise [8–10] and revealed that interplay between nonlinearity and noise results in fascinating noise-induced phenomena including frequency modulation and noise-induced synchronization [11,12].

This extended phase reduction method, however, has found limited applications so far, since the method is applicable only to Gaussian noise. While the noise in the real world often has non-Gaussian statistics, few theories have considered nonlinear systems subject to general non-Gaussian noise, which has forced people to use the Gaussian approximation. In particular, whether the phase description is still valid for oscillators subject to non-Gaussian noise and how quantifiers of the phase dynamics should be amended remains unknown. In this Letter, we develop the phase reduction method for limit-cycle oscillators subject to general, colored and non-Gaussian noise. By correctly evaluating the influence of amplitude perturbations up to second order in the noise strength, we derive the stochastic differential equation of phase, which allows us to study nonlinear oscillations in the real world without the Gaussian approximation. To confirm consistency of the result, we derive closed expressions of quantifiers of the

phase dynamics such as mean frequency, phase diffusion constant, and the Lyapunov exponent. The only limitation we impose is the weakness of the noise. Thus, the obtained results are applicable even when higher order moments of the noise diverge as long as the second order moment is finite and we confirm this fact numerically. As an application of the results, we study a limit-cycle oscillator driven by a phase noise with a finite correlation time and show that amended quantifiers precisely predict resonance between phase and the noise.

We start with the case of a two-dimensional *limit-cycle oscillator* and then extend our results to higher dimensions and multicomponent noise. One can describe the evolution of the system subject to noise in terms of the phase ϕ and the amplitude deviation r from the limit cycle [10,13];

$$\dot{\phi} = \omega + \sigma f(\phi, r)\eta(t), \tag{1}$$

$$\dot{r} = -\lambda r + \sigma g(\phi, r)\eta(t); \tag{2}$$

here ω is the cyclic frequency of unperturbed oscillations; $\lambda := -(\omega/2\pi)\ln\Lambda$ and Λ is the Floquet multiplier of the cycle, i.e., λ is the average amplitude relaxation rate; $\eta(t)$ is a normalized noise; $\sigma \ll 1$ is the noise amplitude; $f(\phi, r)$ and $g(\phi, r)$ are 2π-periodic in ϕ and represent sensitivity of the phase and amplitude, respectively, to noise. The amplitude deviation is nonuniformly scaled so that Eq. (2) is not an approximation, but uniformly valid over the basin of attraction of the limit cycle, as we rigorously show in the supplementary material [13].

We use σ as an expansion parameter; $\phi(t) = \phi_0(t) + \sigma\phi_1(t) + \sigma^2\phi_2(t) + \cdots$, $r(t) = \sigma r_1(t) + \sigma^2 r_2(t) + \cdots$, $f(\phi, r) = f_0(\phi) + f_1(\phi)r + \cdots$, and $g(\phi, r) = g_0(\phi) + g_1(\phi)r + \cdots$. From Eqs. (1) and (2), $\phi_0(t) = \omega t$, $\dot{\phi}_1 = f_0(\phi_0(t))\eta(t)$, and $\dot{r}_1 = -\lambda r_1 + g_0(\phi_0(t))\eta(t)$; the latter two formulae provide

$$\phi_1(t) = \int_{-\infty}^{t} f_0(\phi_0(t_1))\eta(t_1)dt_1, \qquad (3)$$

$$r_1(t) = \int_{0}^{+\infty} g_0(\phi_0(t) - \omega\tau)\eta(t - \tau)e^{-\lambda\tau}d\tau. \qquad (4)$$

Meanwhile, the expansion of Eq. (1) reads

$$\dot\phi = \omega + \sigma f_0(\phi_0)\eta + \sigma^2[f_0'(\phi_0)\phi_1\eta + f_1(\phi_0)r_1\eta] + O(\sigma^3),$$

here prime denotes derivative with respect to ϕ. The right-hand part of the latter equation except for the term proportional to $f_1(\phi)$ is merely the expansion of Eq. (1) with $f(\phi, r)$ replaced by $f(\phi, 0)$. Therefore, we can keep the equation unexpanded with respect to ϕ but add the correction owing to $r_1(t)$;

$$\dot\phi = \omega + \sigma f_0(\phi)\eta(t) + \sigma^2 f_1(\phi_0)r_1\eta(t) + O(\sigma^3).$$

$\sigma^2 f_1(\phi_0)r_1\eta(t)$ is small in comparison to $\sigma f_0(\phi)\eta(t)$, but makes an average contribution of the same order (because $\langle\dot\phi_1\rangle = 0$). Thus, the fluctuating part of this term is not principal and may be omitted;

$$\dot\phi \approx \omega + \sigma f_0(\phi)\eta(t) + \langle\sigma^2 f_1(\phi_0)r_1\eta(t)\rangle.$$

Employing expression (4) for r_1, we obtain

$$\langle f_1(\phi_0)r_1\eta(t)\rangle = f_1(\phi_0(t))\int_{0}^{+\infty} g_0(\phi_0(t) - \omega\tau)$$
$$\times C(\tau)e^{-\lambda\tau}d\tau,$$

where $C(\tau) := \langle\eta(t)\eta(t - \tau)\rangle$ is the noise autocorrelation function. Finally, the reduced phase equation up to the leading contributions reads

$$\dot\phi = \omega + \sigma f_0(\phi)\eta(t)$$
$$+ \frac{\sigma^2}{\omega}f_1(\phi)\int_{0}^{+\infty} g_0(\phi - \psi)C\left(\frac{\psi}{\omega}\right)e^{-(\lambda/\omega)\psi}d\psi. \qquad (5)$$

Here τ is replaced with ψ/ω; the corrections to $\dot\phi$ caused by replacement of ϕ_0 with ϕ in the integrand are $\propto \sigma^3$ and thus negligible.

For Ornstein-Uhlenbeck (OU) noise, $C(\tau) = \gamma\exp(-\gamma|\tau|)$, the reduced phase Eq. (5) takes the form

$$\dot\phi = \omega + \sigma f_0(\phi)\eta(t)$$
$$+ \frac{\sigma^2\gamma}{\omega}f_1(\phi)\int_{0}^{+\infty} g_0(\phi - \psi)e^{-[(\lambda+\gamma)/\omega]\psi}d\psi,$$

which coincides with the one presented in Ref. [10] and implies the corresponding results of Refs. [8,9,14]. While Ref. [9] considers the case of Gaussian noise, a highly stable limit cycle and short noise correlation times and Ref. [10] is limited to the case of OU noise, the present theory includes their results (as special cases) and additionally allows dealing with non-Gaussian noise, arbitrary noise autocorrelation functions (including signals of chaotic oscillators) and arbitrary rate of amplitude relaxation.

The procedure for deriving the reduced phase equation suggests that this equation will provide the correct probability density function for ϕ and mean frequency $\Omega \equiv \langle\dot\phi\rangle$ up to $O(\sigma^2)$;

$$\Omega = \omega + \frac{\sigma^2}{\omega}\left\langle f_0'(\phi)\int_{0}^{+\infty} f_0(\phi - \psi)C\left(\frac{\psi}{\omega}\right)d\psi\right\rangle_\phi$$
$$+ \frac{\sigma^2}{\omega}\left\langle f_1(\phi)\int_{0}^{+\infty} g_0(\phi - \psi)C\left(\frac{\psi}{\omega}\right)e^{-(\lambda/\omega)\psi}d\psi\right\rangle_\phi$$
$$(6)$$

[henceforth, $\langle\cdots\rangle_\phi \equiv (2\pi)^{-1}\int_{0}^{2\pi}\ldots d\phi$]. The noise can either increase or decrease the mean frequency, depending on features of correlation function $C(\tau)$, sensitivity functions, and the cycle stability (e.g., see Fig. 2). However, one should verify whether the more subtle quantities—the phase diffusion constant D and the leading Lyapunov exponent λ_0—can be correctly evaluated from Eq. (5).

The principal contributions to the phase diffusion are readily determined from Eq. (5); indeed,

$$D = \int_{-\infty}^{+\infty}\langle[\dot\phi(t) - \langle\dot\phi\rangle][\dot\phi(t + \tau) - \langle\dot\phi\rangle]\rangle d\tau$$
$$= \sigma^2\int_{-\infty}^{+\infty}\langle\dot\phi_1(t)\dot\phi_1(t + \tau)\rangle d\tau + O(\sigma^4)$$
$$= \frac{\sigma^2}{2\pi}\int_{0}^{2\pi}d\phi\int_{-\infty}^{+\infty}d\tau f_0(\phi)f_0(\phi + \omega\tau)C(\tau) + O(\sigma^4);$$
$$(7)$$

$\dot\phi_1(t)$ [Eq. (3)] is precisely determined by terms accounted in Eq. (5); therefore, Eq. (7) is completely consistent with the reduced phase equation. Interestingly, up to the leading order of accuracy the phase diffusion is not affected by the extra amplitude terms. Thus, for instance, the analytical results and important conclusions of Refs. [15,16] for limit-cycle oscillators subject to weak noise and delayed feedback control remain correct.

For the leading Lyapunov exponent, the situation is more subtle. To deal with it rigorously, we consider a small perturbation ($\alpha = \alpha_0\exp[\mu(t)]$, s) to the solution [$\phi(t)$, $r(t)$] of Eqs. (1) and (2). We have

$$\dot\mu = \sigma(f_0'(\phi(t)) + r(t)f_1'(\phi(t)))\eta(t)$$
$$+ \sigma f_1(\phi(t))\frac{s}{\alpha_0}\eta(t)e^{-\mu},$$
$$\dot s = -\lambda s + \sigma g_0'(\phi(t))\alpha_0 e^\mu\eta(t) + \sigma g_1(\phi(t))s\eta(t),$$

and employ the standard multiscale method adopting $\mu(t) = \mu(t_0, t_2, \ldots)$, $d/dt = \partial/\partial t_0 + \sigma^2\partial/\partial t_2 + \cdots$, etc. After some calculations, one finds the expression for the leading Lyapunov exponent $\lambda_0 := \langle\dot\mu\rangle$ up to $O(\sigma^2)$:

$$\lambda_0 = \frac{\sigma^2}{\omega}\left\langle f_0''(\phi)\int_0^{+\infty} f_0(\phi-\psi)C\left(\frac{\psi}{\omega}\right)d\psi + \frac{\partial}{\partial\phi}\left[f_1(\phi)\right.\right.$$

$$\left.\left. \times \int_0^{+\infty} g_0(\phi-\psi)C\left(\frac{\psi}{\omega}\right)e^{-(\lambda\psi/\omega)}d\psi\right]\right\rangle_\phi + O(\sigma^4)$$

$$= -\frac{\sigma^2}{\omega}\left\langle f_0'(\phi)\int_0^{+\infty} f_0'(\phi-\psi)C\left(\frac{\psi}{\omega}\right)d\psi\right\rangle_\phi + O(\sigma^4),$$
$$(8)$$

which is consistent with the phase equation (5). Note, in the latter equations, the amplitude degree of freedom, which was disregarded in previous works, impacts the instantaneous growth rate of perturbations, but averages out to zero. Thus, on the one hand, our results demonstrate the importance of amplitude degrees of freedom for the stability of response of a general limit-cycle oscillator even in the limit of vanishing noise; on the other hand, its average impact turns out to be zero up to the leading order of accuracy for general noise, proving that analytical calculations and conclusions presented in Refs. [12,16] are valid for real situations. Notice, the negative Lyapunov exponent and its decrease with increase of the noise strength are related to the stability of the noisy system response in sense that it attracts trajectories (the phenomenon is known as noise-induced synchronization), but this does not mean that the response is regular due to the nonzero phase diffusion.

All the results can be extended in a straightforward manner to the case of an N-dimensional dynamical system subject to M-component noise;

$$\dot\phi = \omega + \sum_{\beta=1}^{M}\left[\sigma_\beta f_\beta(\phi,\mathbf{0})\eta_\beta(t) + \sum_{j=1}^{N-1}\frac{\sigma_\beta^2}{\omega}\left(\frac{\partial f_\beta(\phi,\mathbf{r})}{\partial r_j}\right)_{\mathbf{r}=0}\right.$$

$$\left. \times \int_0^{+\infty} g_{\beta,j}(\phi-\psi,\mathbf{0})C_\beta\left(\frac{\psi}{\omega}\right)e^{-(\lambda_j\psi/\omega)}d\psi\right],$$
$$(9)$$

$$D = \sum_{\beta=1}^{M}\frac{\sigma_\beta^2}{\omega}\left\langle f_\beta(\phi,\mathbf{0})\int_{-\infty}^{+\infty} f_\beta(\phi-\psi,\mathbf{0})C_\beta\left(\frac{\psi}{\omega}\right)d\psi\right\rangle_\phi,$$
$$(10)$$

$$\lambda_0 = -\sum_{\beta=1}^{M}\frac{\sigma_\beta^2}{\omega}\left\langle\frac{\partial f_\beta(\phi,\mathbf{0})}{\partial\phi}\int_0^{+\infty}\frac{\partial f_\beta(\phi-\psi,\mathbf{0})}{\partial\phi}\right.$$

$$\left. \times C_\beta\left(\frac{\psi}{\omega}\right)d\psi\right\rangle_\phi.$$
$$(11)$$

Here β indexes noise components, j does the degrees of freedom transversal to the limit cycle.

Now, we address the issue of applicability of our results for noise with diverging higher moments. Although the derived expressions involve only second moments of the noise, one has to check that possible divergence of higher moments does not break the entire expansion and influence Ω, D, and λ_0 in the main order.

For this reason we performed numerical simulation of a Hopf oscillator subject to colored noise $\eta(t)$:

$$\dot A = iA + (\lambda/2)(1-|A|^2)A + \sigma\eta, \qquad (12)$$

$$\dot\eta = \tau_\eta^{-1}[-\eta + s(\eta)\xi(t)], \qquad (13)$$

where A is complex, the noise acts only on $\mathrm{Re}(A)$, $\xi(t)$ is Gaussian white noise: $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$. We consider normalized noises $\eta(t)$ ($\langle\eta^2\rangle = 1$) with three kinds of distribution $V(\eta)$: (1) Gaussian, $V_1(\eta) = (2\pi)^{-1/2}\times\exp(-\eta^2/4)$; (2) exponential, $V_2(\eta) = (1/4)\times\exp(-|\eta|/2)$, which has nonzero but still finite higher cumulants; and (3) fractional rational function, $V_3(\eta) = \pi^{-1}(1+\eta^2)^{-2}$, for which $\langle\eta^{2n}\rangle$ is finite only for $n=1$. These noises are generated with employment of $s_1(\eta) = 1$, $s_2(\eta) = \sqrt{1/4+|\eta|/2}$, and $s_3(\eta) = \sqrt{(1+\eta^2)/3}$ in Eq. (13). For the oscillator (12), one finds $f_0 = -\sin\phi$, $f_1 = -f_0 = \sin\phi$, and $g_0 = \cos\phi$; therefore,

$$\Omega = 1 - \frac{\sigma^2}{2}\int_0^\infty(\sin\psi)(1-e^{-\lambda\psi})C(\psi)d\psi, \qquad (14)$$

$$D = -2\lambda_0 = \sigma^2\int_0^\infty(\cos\psi)C(\psi)d\psi. \qquad (15)$$

For exponential and fractional rational distributions, the correlation function $C(\tau)$ was calculated numerically. In Fig. 1 one can see that the analytical theory is in fairly good agreement with results of numerical simulation both for noises with all moments finite [(b), (c)] and for one with infinite $\langle\eta^4\rangle$ (d). For the latter case the analytical theory is practically no less accurate than for the former ones.
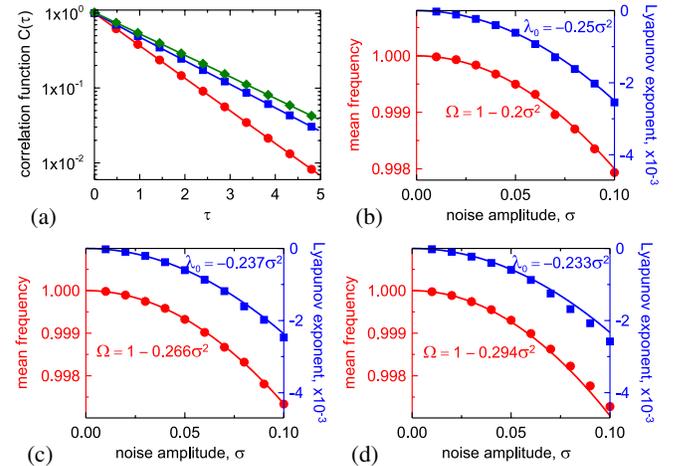

FIG. 1 (color online). Hopf oscillator (12) subject to different noises; here $\tau_\eta = 1$ and $\lambda = 2$. (a) Correlation function $C(\tau)$ for Ornstein-Uhlenbeck noise, which is Gaussian, (red circles) and noises with exponential (blue squares) and fractional rational (green diamonds) distributions. (b)–(d) The numerically calculated mean frequency (red circles) and Lyapunov exponent (blue squares) are in good agreement with Eqs. (14) and (15) (solid lines) for OU noise (b) and noises with exponential (c) and fractional rational (d) distributions.
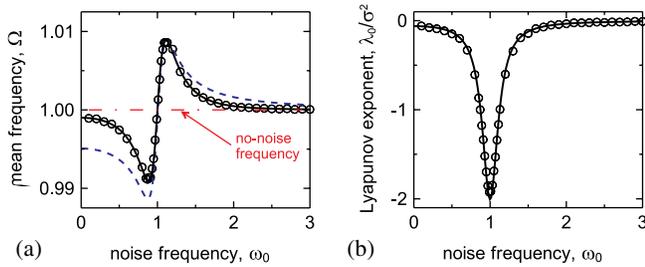
FIG. 2 (color online).   Hopf oscillator (12) subject to phase noise $\eta(t) = \sqrt{2} \cos[\omega_0 t + \sqrt{\gamma} \int^t \xi(t_1) dt_1]$ for $\sigma = 0.1$, $\gamma = 0.125$, $\lambda = 0.4$. Circles: numerical simulation, solid line: analytical theory [Eqs. (14) and (15)], dashed line: analytical theory disregarding the amplitude degree of freedom.

Another important particular opportunity yielded by the theory we developed is the treatment of the effect of the phase noise, $\eta(t) = \sqrt{2} \cos[\omega_0 t + \sqrt{\gamma} \int^t \xi(t_1) dt_1]$. With the noise autocorrelation function $C(\tau) = \cos(\omega_0 \tau) \times \exp(-\gamma|\tau|)$ one can evaluate quantifiers, Ω and λ_0. In Fig. 2 the results of numerical simulation for the Hopf oscillator [Eq. (12)] subject to the phase noise are compared to the analytical theory. Two points are worth emphasizing here: (i) Now we have the phase description for general oscillators subject to noise which is the representative of signals of chaotic and stochastic oscillators. This is important because it provides us with a tool to analytically investigate the synchronizing action of another oscillator (either chaotic or stochastic) on the system under consideration in general. (ii) The amplitude degree of freedom is essential here: in the graph for the frequency (Fig. 2), one can see how the analytical theory neglecting the amplitude perturbations (dashed line) is far from the real observations fairly fitted by the theory we have developed. The most remarkable effects here are observed when the characteristic noise correlation time $2\pi/\omega_0$ is commensurable with the natural oscillation period of the system, that is nonsmall, meanwhile the earlier studies were not able to deal with such a case.

Summarizing, we have derived the reduced phase equation for limit-cycle oscillators subject to general non-Gaussian noise. The derived phase equation correctly provides the mean frequency, the phase diffusion constant, and the Lyapunov exponent. Since the noise-induced shift of the mean frequency means the shift of the resonant frequency for entrainment by external forcing [8,10], our result for mean frequency is immediately relevant for all investigations concerning collective phenomena in networks of coupled oscillators, e.g., [1,4,17], where noise is unavoidably present. In particular, the theory is valid for noise which is the representative of signals of chaotic and stochastic oscillators and thus may provide an accurate analytical tool to investigate their synchronizing action. For the Lyapunov exponent, importance of the amplitude degrees of freedom has been proven, though their average impact on the system stability vanishes in the leading order of accuracy. This implies that the analytical theories in earlier studies on the phase diffusion and the Lyapunov exponent, where the amplitude degree of freedom was disregarded (e.g., [12]), remain generally correct. The theory provides opportunity for analytical investigation of the reliability of neurons [18] and consistency of lasers [19] as well as the quality of clocks, electric generators, lasers, *etc.* for general noise and general limit-cycle oscillators.

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